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6. AUTHOR(S)

I. R. Goodman

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Command, Control and Ocean Surveillance Center (NCCOSC)
RDT&E Division
San Diego, CA 92152-5001

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The purpose of this paper is twofold: First, to clarify Lindley's extension of the "dutch book" argument for probability and related functions over the choice of other possible uncertainty functions. Second, to show that Lindley's conclusions concerning the inadmissibility of possibility functions and Dempster-Shafer functions were patently incorrect in general. All of this is accomplished by, once and for all, placing the problem within a rigorous game theoretic setting.

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21a. NAME OF RESPONSIBLE INDIVIDUAL I. R. Goodman	21b. TELEPHONE (Include Area Code) (619) 553-4014	21c. OFFICE SYMBOL Code 421

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ADMISSIBILITY OF POSSIBILITY FUNCTIONS AND
OTHER NON-PROBABILITY FUNCTIONS IN LINDLEY'S
EXTENSION OF THE DEFINETTI-SAVAGE UNCERTAINTY GAME

I. R. GOODMAN

CODE 421

NAVAL OCEAN SYSTEMS CENTER
SAN DIEGO, CALIFORNIA 92152-5000

Abstract

One of the key problems remaining in the design of an expert system - as well as in Artificial Intelligence models in general - is what uncertainty function or measure is most appropriate to use. Should one choose a Dempster-Shafer, fuzzy set, or classical probability approach, among a myriad number of possibilities?

The purpose of this paper is twofold: First, to clarify Lindley's extension of the "dutch book" argument for probability and related functions over the choice of other possible uncertainty functions. Second, to show that Lindley's conclusions concerning the inadmissibility of possibility functions and Dempster-Shafer functions were patently incorrect in general. All of this is accomplished by, once and for all, placing the problem within a rigorous game theoretic setting.

1. INTRODUCTION

In [1], Bacchus et al. have established certain arguments against the bayesianist practice of assuming, or attempting to demonstrate, that degrees of personal belief must coincide with probability. Their case covers various aspects of bayesian conditional probability updating approaches, including the general static dutch book justification for conditionalization, as well as dynamic updating, reflection, and Carnap's confirmation approach. But, it is only the first topic that this paper wishes to address; the remainder will be left to a future work. It is clear, that despite the invective, Bacchus et al. have not really analyzed in full depth Lindley's results [2]. But, neither has Lindley, despite his additional comments railing against non-probability procedures [3]! In addition, recently, Klir [4] has added to the controversy - part of the Cambridge Debate on Uncertainty, but not published there [5] - by opposing Lindley, not on his own grounds within the setting of Lindley's assumptions, but rather by appealing (rather attractively) to other criteria. There is no question that polemics must be put aside and open unbiased analysis be carried out on this issue.

Consequently, this paper is devoted to re-examining Lindley's contention that probability is essentially the only "admissible" uncertainty function within "rational" context. Since Lindley's argument was couched in seemingly informal, and at times, vague language, the entire problem is restated within a rigorous game-theoretic setting, which is natural to the issue. It becomes clear that Lindley's use of "admissibility" is a much stronger concept than ordinary admissibility; nevertheless, his first set of conclusions that the class of (uniformly - to be explained) admissible uncertainty functions must coincide with some fixed monotone transform of probability remains valid. However, it can be shown that his further conclusions that this implies possibility functions and Dempster-Shafer functions are necessarily inadmissible (in Lindley's uniform sense) in general is wrong. In a word: probability is preserved when a monotone (or other) transform is taken within the argument of the operator, but not when it is composed from the outside with probability. Specifically, it is shown that there are large classes of fuzzy set membership functions and their t-conorm possibility extensions (see [6]) which are indeed

admissible completely within Lindley's sense. Furthermore, there is at least one type of non-admissible possibility measure - Zadeh's max-possibility - which under many circumstances is the uniform limit of admissible possibility functions. In addition, not all Dempster-Shafer belief functions are inadmissible - the positive powers (exceeding unity) of probabilities are all admissible Dempster-Shafer belief functions.

Finally, it is hoped that by putting the dutch book problem within a purely game-theoretic setting, additional properties of this game - such as game value, or upper game value, least favorable priors, minimax uncertainty functions, etc. - can be obtained and utilized. Certainly, there should be room for extending the game to other types of general loss functions, as well as to include epistemic considerations not touched upon so far. Because of space limitations, a number of topics have been omitted here and are considered in some detail in [13].

2. NOTATION AND DEFINITIONS

Throughout the remainder of this paper, unless otherwise specified, suppose the following obtains:

Ω is a fixed nonvacuous universal set of points $\omega \in \Omega$, with $R \subseteq P(R)$, a boolean algebra with events indicated by $a, a_1, a_2, \dots, b, b_1, b_2, \dots \in R$, and having the usual boolean operators: \cdot , conjunction / intersection; \vee , disjunction / union; and $()'$, negation / complement. In addition, \leq represents the usual partial (lattice) order over R corresponding to subevent ordering. Conditional events are denoted typically as $(a|b)$ (a being consequent and b antecedent of the event), where each can be interpreted in at least three equivalent ways: principal ideal cosets, closed intervals of events, three-valued logically via DeFinetti's indicator function. (See [7],[8] for expositions on conditional events.) Taking the third approach, one writes

$$\phi(a|b)(\omega) \stackrel{d}{=} \begin{cases} 1 & \text{iff } (a|b) \text{ occurs at } \omega \text{ iff } \omega \in a \cdot b \\ 0 & \text{iff } (a|b) \text{ does not occur at } \omega \text{ iff } \omega \in a' \cdot b \\ u & \text{iff } (a|b)'s \text{ occurrence at } \omega \text{ is undetermined} \\ & \text{iff } \omega \in b'. \end{cases} \quad (1)$$

In conjunction with the regions for determining the conditional event indicator function, define functions w_j , $j \in K_0$, where

$$K_0 \stackrel{d}{=} \{0, u, 1\}, \quad (2)$$

and

$$w_1(a|b) \stackrel{d}{=} a \cdot b, \quad w_0(a|b) \stackrel{d}{=} a' \cdot b, \quad w_u(a|b) \stackrel{d}{=} b'. \quad (3)$$

Although, conditional events will play some role in the development of the uncertainty game, many of their properties are not needed here. However, it should be noted that the characterization in (1) is sufficient to show

$$(a|b) = (a \cdot b|b), \quad (a|\Omega) = a, \quad (4)$$

to which one adds the natural evaluation for any given probability (always assumed here no stronger than being finitely additive) $p: R \rightarrow [0,1]$, $[0,1]$ being the unit interval,

$$p((a|b)) = p(a|b) \stackrel{d}{=} p(a \cdot b)/p(b) \quad ; \text{ for } p(b) > 0, \quad (5)$$

ordinary conditional probability. Denote the space of all conditional events $(a|b)$ for $a, b \in R$ arbitrary, as $(R|R)$. Hence, by (4), $R \subseteq (R|R)$.

Introduce also the following multivariable notation, for any given positive integer n and $a_i, b_i \in R$:

$$\left. \begin{aligned} \underline{a} &\triangleq (a_1, \dots, a_n); \quad \underline{b} \triangleq (b_1, \dots, b_n); \quad (\underline{a}|\underline{b}) \triangleq ((a_1|b_1), \dots, (a_n|b_n)) \in (R|R)^n; \quad \phi(\underline{a}|\underline{b})(\omega) \triangleq \\ &(\phi(a_1|b_1)(\omega), \dots, \phi(a_n|b_n)(\omega)) \in K_0^n; \quad \underline{j} \triangleq (j_1, \dots, j_n) \in K_0^n; \quad \underline{w}(\underline{a}|\underline{b}) \triangleq \left\{ \prod_{i=1}^n w_{j_i}(a_i|b_i) \right\} \\ &\omega(\underline{a}|\underline{b}) \triangleq \{w_{\underline{j}}(\underline{a}|\underline{b}): \underline{j} \in K_0^n\}, \end{aligned} \right\} \quad (6)$$

noting that since $\{w_{\underline{j}}(\underline{a}|\underline{b}): \underline{j} \in K_0^n\}$ is a (disjoint, exhaustive) partitioning of Ω , so is $\omega(\underline{a}|\underline{b})$.

Call $f: [0,1] \times K_0 \rightarrow \mathbb{R}$ (real line) a betting or score function iff $f(\cdot, j)$ is continuously differentiable over $[0,1]$, denoting $f'(s, j) = \partial f(s, j) / \partial s$, $s \in [0,1]$, $j=0,1$, with $f(\cdot, 0)$ strictly increasing and $f(\cdot, 1)$ strictly decreasing over $[0,1]$, so that

$$0 = f'(j, j), \quad j=0,1; \quad f(\cdot, u) = 0. \quad (7)$$

A prime example of a betting function is f_0 , essentially the case treated in e.g. [9] leading to probability, where

$$f_0(s, j) \triangleq \begin{cases} \lambda_j \cdot (s-j)^2, & j=0,1, \quad s \in [0,1], \\ 0, & j=u, \quad s \in [0,1], \end{cases} \quad (8)$$

where λ_j are fixed real constants.

For any $\underline{s} \in [0,1]^n$, $\underline{t} \in K_0^n$, define

$$\hat{f}(\underline{s}, \underline{t}) \triangleq \sum_{i=1}^n f(s_i, t_i). \quad (9)$$

Then, define the two player zero-sum uncertainty game

$$G_f \triangleq G(N, U; \text{loss}_f) \quad (10)$$

as follows: Player 1 or Nature, has as its space of pure strategies

$$N \triangleq \{\phi(\underline{a}|\underline{b})(\omega): \omega \in \Omega, (\underline{a}|\underline{b}) \in (R|R)^n, n=1,2,\dots\}, \quad (11)$$

each $\phi(\underline{a}|\underline{b})(\omega)$ being a possible outcome/no outcome/undetermined outcome-combination, determined by finite sequence of conditional events $\{\underline{a}|\underline{b}\}$ and ω . Player 2, or Decision-maker, has as its space of pure strategies

$$U \triangleq [0,1](R|R) = \{q: q: (R|R) \rightarrow [0,1]\}, \quad (12)$$

where each q in (12) is called an uncertainty function, $q(\underline{a}|\underline{b})$, for any $(\underline{a}|\underline{b}) \in (R|R)$, representing the uncertainty of $(\underline{a}|\underline{b})$ occurring, i.e., that $\omega \in \Omega$, unknown to Player 2, is such that $\phi(\underline{a}|\underline{b})(\omega) = 1$, i.e., $\omega \in \underline{a} \cdot \underline{b}$. Thus, U certainly contains all conditional or unconditional possibility functions, probability measures (finitely additive), Dempster-Shafer functions, etc.

Also, for each betting function f , define loss function $\text{loss}_f: N \times U \rightarrow \mathbb{R}$, by

$$\text{loss}_f(q, \phi(\underline{a}|\underline{b})(\omega)) \triangleq \hat{f}(q(\underline{a}|\underline{b}), \phi(\underline{a}|\underline{b})(\omega)); \quad q(\underline{a}|\underline{b}) \triangleq (q(a_1|b_1), \dots, q(a_n|b_n)), \text{ all } (\underline{a}|\underline{b}), \omega, q. \quad (13)$$

Clearly, (13) implies the expansion, for any $(\underline{a}|\underline{b}), \omega, q$:

$$\text{loss}_f(q, \phi(\underline{a}|\underline{b})(\omega)) = \sum_{j=0}^1 \left(\sum_{\{i: 1 \leq i \leq n \text{ \& } \omega \in w_{\underline{j}}(\underline{a}|\underline{b})\}} f(q(a_i|b_i), j) \right). \quad (14)$$

and for any random variable W over Ω with prior prob. distribution F , the expected loss is

$$\rho(q, F; \underline{a}|\underline{b}) \triangleq E_W(\text{loss}_f(q, \phi(\underline{a}|\underline{b})(W))) = \sum_{i \in M(\underline{b}, F)} \Pr(W \in b_i) \cdot \sum_{j=0}^1 F_j(a_i|b_i) \cdot f(q(a_i|b_i), j), \quad (15)$$

where

$$F_j(a_i|b_i) \stackrel{d}{=} \Pr(W \in w_j(a_i|b_i) | W \in b_i), j=0,1, i=1,2,\dots; M(\underline{b}, F) \stackrel{d}{=} \{(i:1 \leq i \leq n \text{ \& } \Pr(W \in b_i) > 0)\}. \quad (16)$$

The loss in (13) can be interpreted as the amount incurred by the Decision-maker when betting function f is agreed to (by players 1 and 2 or outside referee), finite sequence of (conditional) events $(a|b)$ is considered and uncertainty function q is chosen by the Decision-maker to apply to $(a|b)$, and in reality, unknown to the latter, event outcome mechanism is at $\omega \in \Omega$; similarly, for the expected loss relative to ω being assigned a random variable W .

Define also, for each $(a|b) \in (R|R)^n$, the subgame $G_f(a|b)$ of G_f , where N is reduced to $N(a|b)$ and U to $U(a|b)$, with loss_f similarly restricted,

$$G_f(a|b) \stackrel{d}{=} G(N(a|b), U(a|b); \text{loss}_f); N(a|b) \stackrel{d}{=} \{\phi(a|b)(\omega) : \omega \in \Omega\}; U(a|b) \stackrel{d}{=} [0,1]^{(a|b)}. \quad (17)$$

Also, by using (4), one can consider the unconditional subgame $G_f^o(a)$ of G_f , where

$$G_f^o(a) \stackrel{d}{=} G_f(a|\underline{\Omega}) = G(N(a), U(a); \text{loss}_f), \quad (18)$$

where $N(a), U(a)$ are limited to events of a and similarly for loss_f . One can also define the full unconditional subgame G_f^o of G_f , where $N \stackrel{d}{=} \{\phi(a)(\omega) : \omega \in \Omega\}$ replaces N and $U = [0,1]^R$ replaces U , etc.

With the above established, define the uniform counterparts of two basic concepts of game theory, keeping in mind the unconditional counterparts to the definitions:

(i) For any given uncertainty function $q_0 \in U$, call q_0 uniformly bayes relative to random variable W over Ω with prior prob. dist. F , for game G_f , iff q_0 is bayes relative to W for each subgame $G_f(a|b)$, for all $(a|b) \in (R|R)^n$, $n=1,2,\dots$. Hence,

$$\rho(q_0, F; (a|b)) = \inf_{q \in U(a|b)} \rho(q; F; (a|b)); \text{ all } (a|b). \quad (19)$$

Use the notation that $q_0 = q_{F, (a|b), f}$.

(ii) For any given $q_0 \in U$, call q_0 uniformly admissible for G_f iff q_0 is admissible for each subgame $G_f(a|b)$, for each $(a|b) \in (R|R)^n$, $n=1,2,\dots$. Hence,

$$\left\{ \begin{aligned} & \text{all } (a|b) \text{ (not true there exists } q_{(a|b), f} \in U) \\ & (\text{loss}_f(q_{(a|b), f}, \phi(a|b)(\omega)) \leq \text{loss}_f(q_0, \phi(a|b)(\omega)), \text{ all } \omega \in \Omega, \\ & \text{with strict inequality holding for at least some } \omega \in \Omega) \end{aligned} \right\}. \quad (20)$$

Finally, for each betting function f , define nondecreasing continuous transform $P_f: [0,1] \rightarrow [0,1]$, with $P_f(j) = j$, $j=0,1$, where, for all $s \in [0,1]$,

$$P_f(s) \stackrel{d}{=} 1/(1-Q_f(s)); Q_f(s) \stackrel{d}{=} f'(s,1)/f'(s,0). \quad (21)$$

If $f''(s,j)$ exists, $j=0,1$, with $Q'(s) > 0$, for all $s \in [0,1]$, then P_f will be strictly increasing over $[0,1]$. In particular, this is guaranteed, if $f(s,j)$ is convex in s , for all $s \in [0,1]$.

3. THE BASIC RESULTS

Theorem 1. Properties of the subgames $G_f(a|b)$.

Let f be any betting function, n any positive integer, and $(a|b) \in (R|R)^n$ arbitrary. Then:

- (i) $G_f(a|b)$ is equivalent to an S-game, i.e., player 1's space $M(a|b)$, or equivalently $\omega_f(a|b)$ (with cardinality $\leq 3^n$) is finite. Player 2's space in the usual product topology is compact, as well as being convex and loss_f is continuous. Thus, $\{\text{loss}_f(q, \phi(a|b)) : q \in U(a|b)\}$ is a bounded compact risk set.
- (ii) As a consequence of (i), the classical assumptions of Blackwell & Girshick ([10], section 5.2) and Wald ([11], Chapter 3) are all satisfied. (See also [12], Chapters 1, 2.), showing $G_f(a|b)$ has a game value; a least favorable prior distribution over Ω ; for which a minimax uncertainty measure exists as bayes relative to f ; and the class of all bayes uncertainty functions forms a complete class, among other properties.
- (iii) In particular, for any random variable W over Ω with distribution F , the bayes uncertainty function $q_{F, (a|b), f}$ satisfies the relation

$$P_{f, (a|b), f}(q_{F, (a|b), f}(a_i|b_i)) = F_1(a_i|b_i), \quad \text{all } i \in M(b, F). \quad (22)$$

and, assuming P_f is strictly increasing over $[0,1]$, the corresponding minimal expected loss is

$$\rho(q_{F, (a|b), f}, F; (a|b)) = \sum_{i \in M(b, F)} \Pr(W=b_i) \cdot \sum_{j=0}^1 F_j(a_i|b_i) \cdot f(P_f^{-1}(F_1(a_i|b_i)), j). \quad (23)$$

Proof: (i) and (ii) are self-evident. For (iii), apply the usual differentiation technique to (15) relative to variable $q(a_i|b_i)$. (23) is a result of substituting (22) in (15). ■

Remark. Eq.(23) appears promising for developing closed form expressions for game value, minimax uncertainty function, and least favorable prior, in view of Theorem 1. ■

Theorem 2. Lindley's first basic result restated and reproved ([2], Theorem 2).

For any choice of betting function such that P_f is strictly increasing over $[0,1]$ and any given uncertainty function $q \in U$, the following statements are equivalent:

- (i) q is uniformly admissible for G_f .
- (ii) q is uniformly bayes for G_f .
- (iii) $P_f \circ q$ is the conditional probability extension of some probability over R .

Proof: (ii) implies (i): By the form of bayes uncertainty functions given in Theorem 1, eq.(22), each is unique for subgame $G_f(a|b)$, and hence admissible.

(iii) implies (ii): Retrace the steps of proof of Theorem 1(iii), noting $P_f \circ q$ is a probability over $\omega(a|b)$.

(i) implies (iii): Note first that (i) implies q is admissible for $G_f(a|b)$ for all $(a|b)$ satisfying two classes: (I): $(a|b) = ((a|b), (a'|b))$ ($n=2$), and (II): $(a|b) = ((a|b), b, a \cdot b)$ ($n=3$), for all $a, b \in R$. In turn, use the fact that admissibility implies weak local admissibility (see [13] for complete details), so that sketching the basic properties, eq.(20) yields differential of $\text{loss}_f(q, \phi(a|b))$ with respect to $dq(a|b)$ which cannot be non-positive for all components of $dq(a|b)$ and negative for some one. Equivalently, this means one cannot have in the same ordering sense as above

$$J_f(q, \phi(a|b)) \cdot dq(a|b) \leq 0; \quad J_f(q, \phi(a|b)) \not\leq \partial \text{loss}_f(q, \phi(a|b)) / \partial q(a|b), \quad (24)$$

for all $(a|b)$ of type (I) or (II).

Eq.(24), in turn translates to the conditions $\det(J_f) = 0$, for appropriate rank reductions for cases (I) and (II). This yields:

$$\text{Case (I): } \det \begin{bmatrix} f'(q(a|b), 1) & f'(q(a'|b), 0) \\ f'(q(a|b), 0) & f'(q(a'|b), 1) \end{bmatrix} = 0, \text{ implying } P_f(q(a|b)) + P_f(q(a'|b)) = 1, \quad (25)$$

for all $a, b \in R$.

$$\text{Case (II): } \det \begin{bmatrix} f'(q(a|b), 1) & f'(q(b), 1) & f'(q(a \cdot b), 1) \\ f'(q(a|b), 0) & f'(q(b), 1) & f'(q(a \cdot b), 0) \\ 0 & f'(q(b), 0) & f'(q(a \cdot b), 0) \end{bmatrix} = 0, \quad (26)$$

implying

$$P_f(q(a|b)) \cdot P_f(q(b)) = P_f(q(a \cdot b)), \quad (27)$$

for all $a, b \in R$.

Finally, combining (25) and (27) shows (iii). ■

Remark. The proof technique for (i) implies (iii) is originally due to DeFinetti [9]. By replacing in that proof Cases (I) and (II) by simply

Case(III): $(a|b) = (a, c, avc)$, $a \cdot c = 0$, then one obtains

$$\text{Case(III): } \det \begin{bmatrix} f'(q(a), 1) & f'(q(c), 1) & f'(q(avc), 1) \\ f'(q(a), 0) & f'(q(c), 1) & f'(q(avc), 1) \\ f'(q(a), 0) & f'(q(c), 0) & f'(q(avc), 1) \end{bmatrix} = 0, \quad (28)$$

implying

$$P_f(q(avc)) = P_f(q(a)) + P_f(q(c)), \text{ all } a \cdot c = 0. \quad (29)$$

This proves $P_f \circ q$ is an unconditional probability over R , which together with the remainder of the proof of Theorem 2 modified for G_f remains valid. Thus, Theorem 2 holds for the unconditional subgame G_f , with all statements appropriately made into unconditional ones. ■

Next, call uncertainty function q general uniformly[unconditional] admissible iff there exists betting function f such that q is uniform [unconditional] admissible for $G_f [G_f]$.

Theorem 3. Lindley's second basic result restated ([2], Theorem 2 and Lemma 5).

Let $q \in U$ be any uncertainty function. Then, the following statements are equivalent:

(i) q is general uniformly[unconditional] admissible.

(ii) There exists a continuous increasing transform $P: [0, 1] \rightarrow [0, 1]$ with $P(j)=j$, $j=0, 1$, such that $P \circ q$ is a conditional probability [unconditional probability].

Proof: Immediate from the fact that as f runs over all possible betting functions, P_f runs over all P 's as above, in conjunction with Theorem 2. ■

Remark. It was Lindley's contention that because of Theorem 3's characterization of (general uniform)admissible uncertainty functions, the non-probability-appearing possibility functions as well as Dempster-Shafer and other uncertainty functions could not possibly be (general uniform) admissible. But the class of all monotone increasing continuous etc. transforms on all conditional [or on all unconditional] probabilities is surprisingly rich and goes beyond just probability (when the transform is the identity map). This is shown next.

In order to present the next results, the concepts of t -conorms and t -possibility extensions of fuzzy sets (or fuzzy set membership functions, equivalently) are reviewed. For further details, see e.g. Goodman & Nguyen ([6], Chapter 2).

A t -conorm is a semi-group-like binary operation $t: [0,1]^2 \rightarrow [0,1]$. Specifically, t is associative, commutative, nondecreasing in each argument with $t(x,0)=x$ and $t(x,1)=1$, for all $x \in [0,1]$. If also, $t(x,x) > x$, all $0 < x < 1$, call t archimedean. The following theorem is extremely important in the next development:

Theorem 4. (Ling's Theorem [14] .)

(i) Let t be any archimedean t -conorm. Then, there exists a continuous increasing function $g_t: [0,1] \rightarrow [0, \infty]$ with $g(0)=0$ such that

$$t(x,y) = g_t^{-1}(\min(g_t(x)+g_t(y), g_t(1))), \text{ all } x,y \in [0,1]. \quad (30)$$

(ii) Conversely, if g is any function satisfying the same properties as g_t above eq.(30), call g a generator, and any function t defined as in (30), g replacing g_t , yields a legitimate t -conorm which is archimedean.

Remarks and definitions. By utilizing associativity, any t -conorm can be naturally extended to any finite- and by limits, to countably infinite- number of arguments, where, by convention $t(x) \doteq x$ (i.e., identity) for all $x \in [0,1]$. By Ling's Theorem, any archimedean t -conorm's multi-argument extension has the simple form

$$t(x_1, \dots, x_n) = g_t^{-1}(\min(g_t(x_1) + \dots + g_t(x_n), g_t(1))), \text{ all } x_i \in [0,1]. \quad (31)$$

\max is an example of a non-archimedean t -conorm, while probsum , t_p , bndsum are examples of archimedean t -conorms, among an infinitude of such (again, see [6]), where using multivariable notation

$$\underline{x} = (x_1, \dots, x_n) \in [0,1]^n; \text{sum}(\underline{x}) \doteq x_1 + \dots + x_n; \underline{1-x} = (1-x_1, \dots, 1-x_n); \text{prod}(\underline{y}) \doteq y_1 \cdots y_n, \quad (32)$$

all $x_i, y_i \in [0,1]$, etc.

$$\text{probsum}(\underline{x}) = 1 - \text{prod}(\underline{1-x}); g_{\text{probsum}}(x) = -\log(1-x), \text{ all } x \in [0,1]; \quad (33)$$

$$t_p(\underline{x}) = \min(\text{sum}(\underline{x}^p), 1)^{1/p}; g_t(x) = x^p, \text{ all } x \in [0,1]; p \geq 1. \quad (34)$$

$$\text{bndsum}(\underline{x}) = \min(\text{sum}(\underline{x}), 1); g_{\text{bndsum}}(x) = x \text{ (identity)}, \text{ all } x \in [0,1]. \quad (35)$$

Finally, supposing as usual $R \subseteq P(\Omega)$, but now that Ω is finite or countably infinite in cardinality, let $f: \Omega \rightarrow [0,1]$ be any fuzzy set (membership function) on Ω and t any t -conorm. Then, the t -possibility extension of f is $\delta_t: R \rightarrow [0,1]$ given by

$$\delta_t(\{a\}) \doteq f(a), a \in \Omega; \delta_t(a) \doteq t(f(\omega): \omega \in a), \text{ all } a \in R. \quad (36)$$

t -possibility extensions are all thus in \mathcal{U} and play a lead role in fuzzy set theory as well as in random set theory in relating fuzzy sets and possibilities to probabilities (again, see [6]). Note that when t is an archimedean t -conorm with generator g_t , (36) becomes

$$\text{all } a \in R: \delta_t(a) = g_t^{-1}(\min(\text{sum}(f(a)), g_t(1))); f(a) \doteq (f(\omega): \omega \in a). \quad (37)$$

Theorem 5. First refutation of Lindley's conclusions

Let $q \in \mathcal{U}$ be any unconditional uncertainty function and, as above (from now on) Ω is finite or countably infinite. Then, the following are equivalent:

- (i) q is general uniformly unconditional admissible.
- (ii) q is the t -possibility extension of some fuzzy set $f: \Omega \rightarrow [0,1]$, where t is archimedean with generator g_t such that $g_t \circ f$ is a probability function over Ω with $g_t(1)=1$.

Proof: (i) implies (ii): Let t have generator $g_t \doteq P_f$. (ii) implies (i): Find betting function f so that $P_f = g_t$. (Both proofs of course use Theorem 3.)

Theorem 6. Second refutation of Lindley.

Let $f: \Omega \rightarrow [0,1]$ be any fuzzy set with $0 < \sup(f(\Omega)) < 1$. Then, the following are equivalent:

- (i) There is an archimedean t-conorm t such that $f_t \in \mathcal{U}$ is general uncond. admissible
- (ii) The level sets of f are all finite. That is, $f^{-1}[s,1]$ is finite, $0 < s < 1$.

Proof: (i) implies (ii): $f^{-1}[s,1]$, for some s being infinite violates Theorem 5(ii).

(ii) implies (i): From Theorem 5, need only construct generator g satisfying Theorem 5(ii). This long construction is given in [13], proof of Theorem 5.2.2. ■

Remark. The t-conorms in (32)-(35) can be used to satisfy Theorem 5(ii) for appropriate choice of fuzzy set f . When Ω is finite, Theorem 6(ii) is satisfied.

If fuzzy set f satisfies hypothesis of Theorem 6, $\sum(f(\Omega)) \leq 1$, and

$$\text{card}(f^{-1}[1/n, 1/(n-1)]) \leq \kappa_j \cdot n^{\kappa_2}, n=2,3,\dots; \kappa_j > 0 \text{ constants, } j=1,2, \quad (38)$$

then, for any fixed positive integer n_0 , the following limit holds uniformly in a :

$$\lim_{p \rightarrow \infty} (f_{t_p}(a)) = f_{\max}(a); \text{ all } a \in R, \text{ card}(a) \leq n_0, a \cdot f^{-1}(\sup(f(\Omega))) = 0. \quad (39)$$

Thus, Zadeh's max possibility extension under general circumstances is the uniform limit of general uniformly unconditional admissible t-possibility extensions of fuzzy sets.

Finally, it is stated without proof (see [13], section 6) that the class of all r^{th} powers of any probability measures (conditional or unconditional), for any $r \geq 1$, consists of general uniformly admissible Dempster-Shafer belief functions. ■

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